Python: recursion

Intro. to computing

`https://mandarm.github.io/computing.html`

Recursion

A recursive function is a function that calls itself.

- The task should be decomposable into sub-tasks that are smaller, but otherwise identical in structure to the original problem.
- The simplest sub-tasks (called the base case) should be (easily) solvable directly, i.e., without decomposing it into similar sub-problems.

Recursive vs. iterative implementations

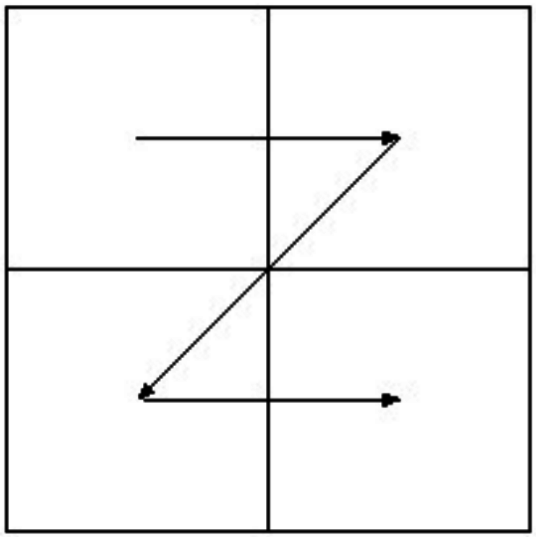
```
def factorial(n) :
    if n < 0 : return -1

    prod = 1
    while n > 0 :
        prod *= n
        n = n - 1
    return prod
```

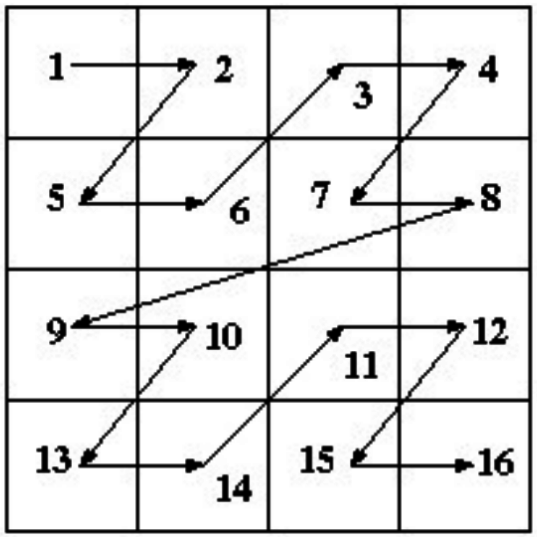
```
def factorial(n) : # Recursive implementation

    if n < 0 : return -1        # Error condition
    if n == 0 : return 1        # Base case
    return n * factorial(n-1)   # Recursive call
```

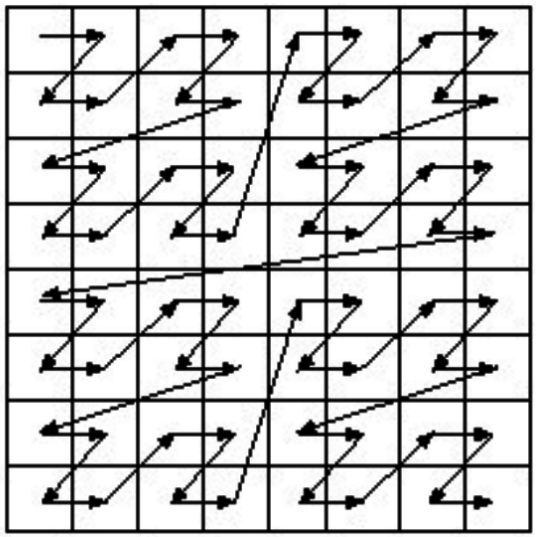
Z curve

Problem statement

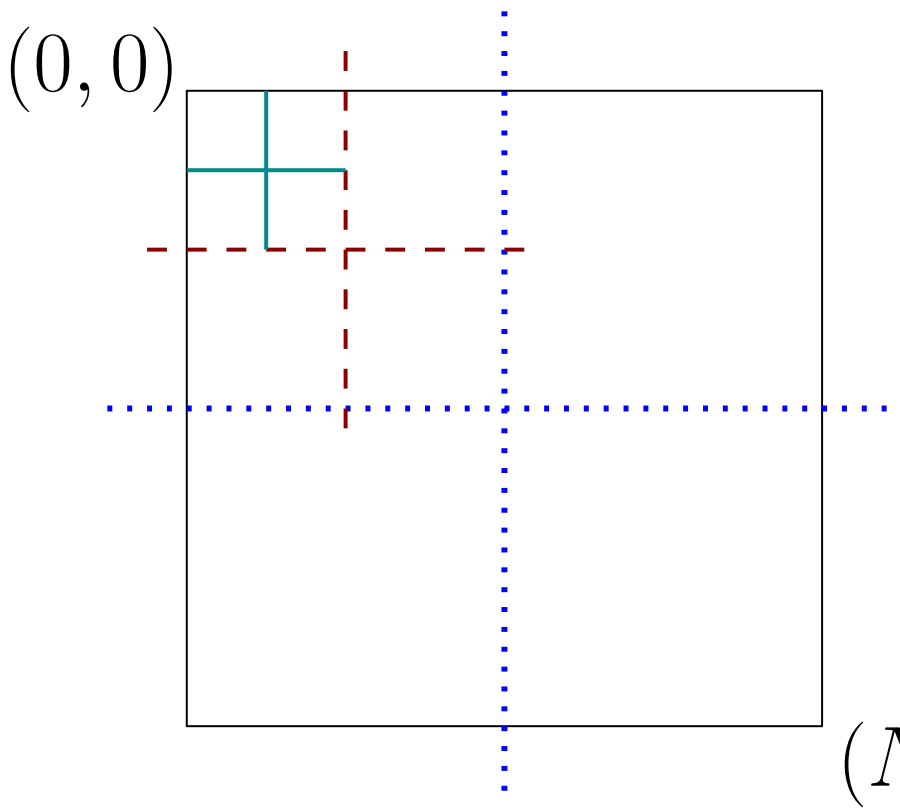
Consider a 2-D matrix of size $2^m \times 2^m$. The entries of the matrix are, in row-major order, $1, 2, 3, \ldots, 2^{2m}$. Print the entries of the matrix in Z-curve order (as shown in the picture below).







Z curve: structure

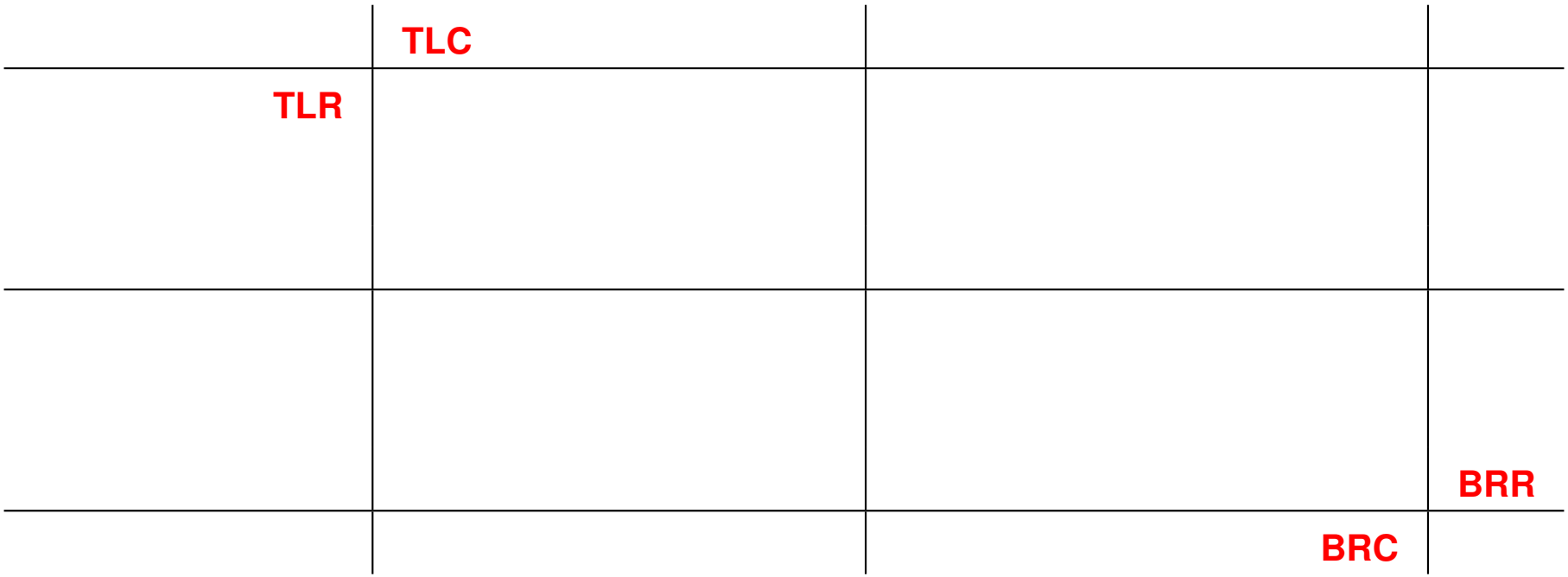


Problem structure:

- *Base case:* single element
- Recursive structure:
  - break given square into 4 sub-squares
  - process squares in Z-curve order

Z curve: code

```
def z_curve(matrix, top_left_row, top_left_column, bottom_right_row,
     bottom_right_column) :

    # Base case
    if top_left_row == bottom_right_row and \
       top_left_column == bottom_right_column :
        print(f"{matrix[top_left_row][top_left_column]}", end=" ")
        return
```



Z curve: code

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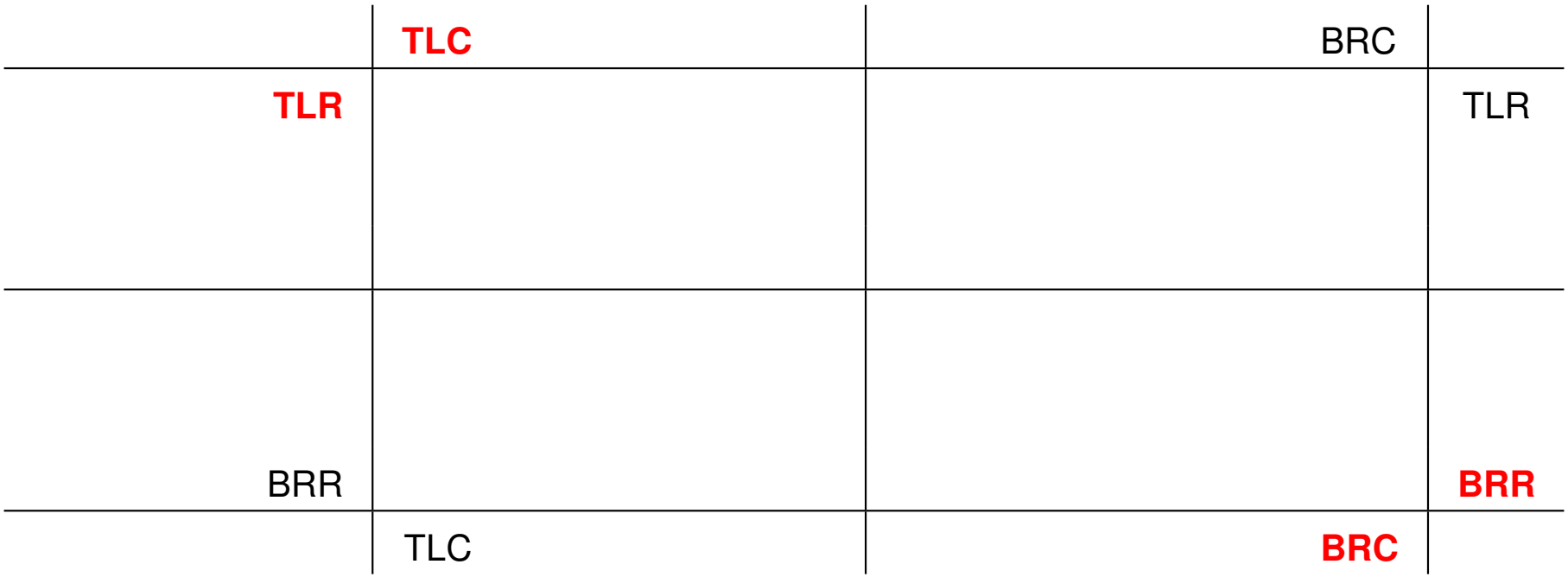

Z curve: code

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def z_curve(matrix, top_left_row, top_left_column, bottom_right_row,
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        print(f"{matrix[top_left_row][top_left_column]}", end=" ")
        return
```

	TLC (TLC + BRC)//2	(TLC + BRC)//2 + 1 BRC	
TLR			TLR
BRR			**BRR**
	TLC	**BRC**	

Z curve: code

```
def z_curve(matrix, top_left_row, top_left_column, bottom_right_row,
     bottom_right_column) :

    # Base case
    if top_left_row == bottom_right_row and \
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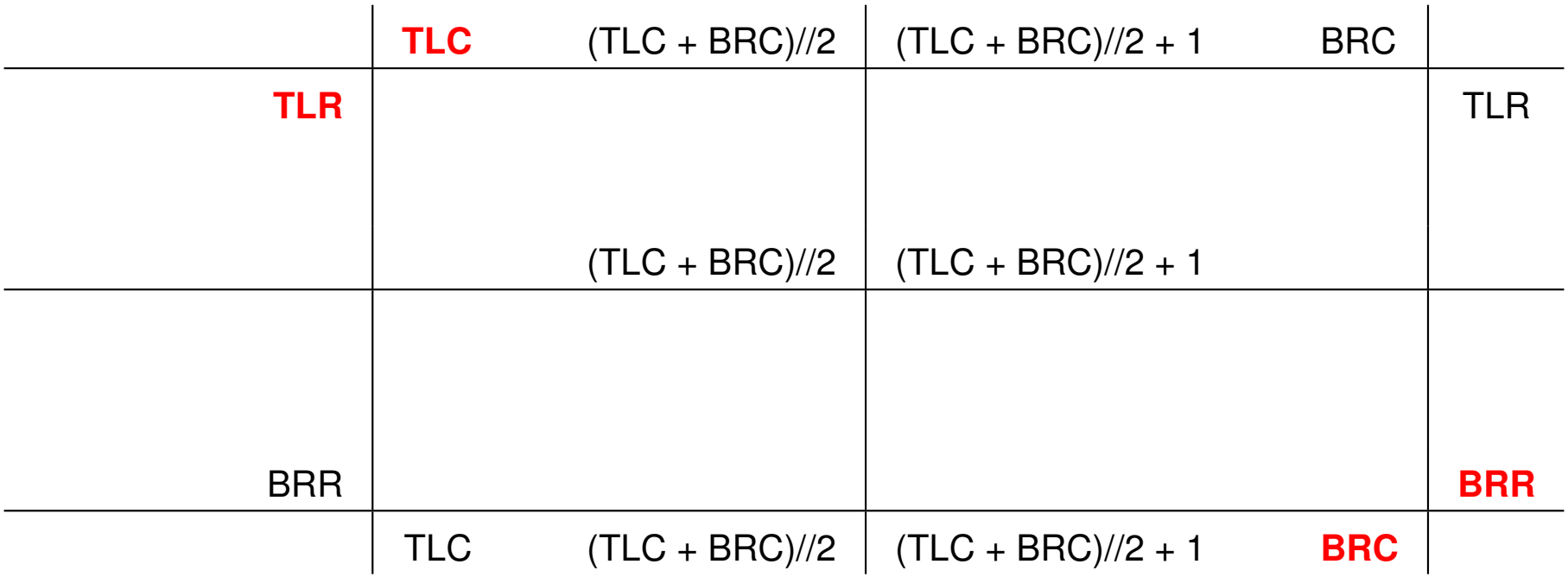


Z curve: code

```
def z_curve(matrix, top_left_row, top_left_column, bottom_right_row,
     bottom_right_column) :

    # Base case
    if top_left_row == bottom_right_row and \
       top_left_column == bottom_right_column :
        print(f"{matrix[top_left_row][top_left_column]}", end=" ")
        return
```

	TLC	(TLC + BRC)//2	(TLC + BRC)//2 + 1 BRC	
TLR				TLR
(TLR + BRR)//2		(TLC + BRC)//2	(TLC + BRC)//2 + 1	
(TLR + BRR)//2 + 1				
BRR				**BRR**
	TLC	(TLC + BRC)//2	(TLC + BRC)//2 + 1 **BRC**	

Z curve: code

```
    # Recurse

    # upper-left sub-square
    z_curve(matrix,
            top_left_row,
            top_left_column,
            (top_left_row + bottom_right_row)//2,
            (top_left_column + bottom_right_column)//2)

    # upper-right sub-square
    z_curve(matrix,
            top_left_row,
            (top_left_column + bottom_right_column)//2 + 1,
            (top_left_row + bottom_right_row)//2,
            bottom_right_column)

    # lower-left sub-square
    z_curve(matrix,
            (top_left_row + bottom_right_row)//2 + 1,
            top_left_column,
            bottom_right_row,
            (top_left_column + bottom_right_column)//2)
```

Z curve: code

```
    # lower-right sub-square
    z_curve(matrix,
            (top_left_row + bottom_right_row)//2 + 1,
            (top_left_column + bottom_right_column)//2 + 1,
            bottom_right_row,
            bottom_right_column)
    return
```

```
N = 2**2
i = 1; j = 0
A = [ [ x for x in range(j*N+i, j*N+i+N) ] for j in range(N) ]
z_curve(A,0,0,N-1,N-1); print()
```

Permutations

Algorithm

To generate all permutations of $1, 2, 3, \ldots, n$, do the following:

1. Generate all permutations of $2, 3, \ldots, n$.
2. For each of the permutations generated above
   (1) add 1 in the beginning;
   (2) add 1 in the 2nd position;
   ...
   (n) add 1 at the end.

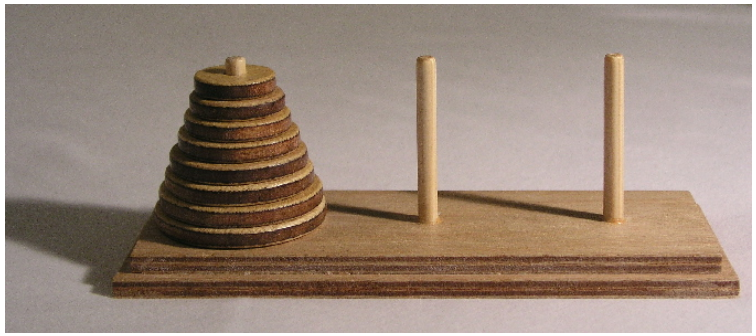
Permutations: code

```
def permute(A) :
    n = len(A)
    if n < 2: return [ A ]

    # Need recursion for n >=2
    permlist = [ ]
    for p in permute(A[1:]) :
        for position in range(n) :
            permlist.append(p[0:position] + [ A[0] ] + p[position:])
    return permlist
```

Exercises – I

1. Towers of Hanoi: see
   `https://en.wikipedia.org/wiki/Tower_of_Hanoi`



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2. Write a recursive function with prototype `int C(int n, int r);` to compute the binomial coefficient using the following definition:

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$$

   Supply appropriate boundary conditions.

Exercises – II

3. Define a function $G(n)$ as:

$$G(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ G(n-1) + G(n-2) + G(n-3) & \text{if } n \geq 3 \end{cases}$$

Write recursive **and** iterative (i.e., non-recursive) functions to compute $G(n)$.

Exercises – III

4. What does the following function compute?

C

```c
int f ( int n )
{
   int s = 0;
   while (n--) s += 1 + f(n);
   return s;
}
```

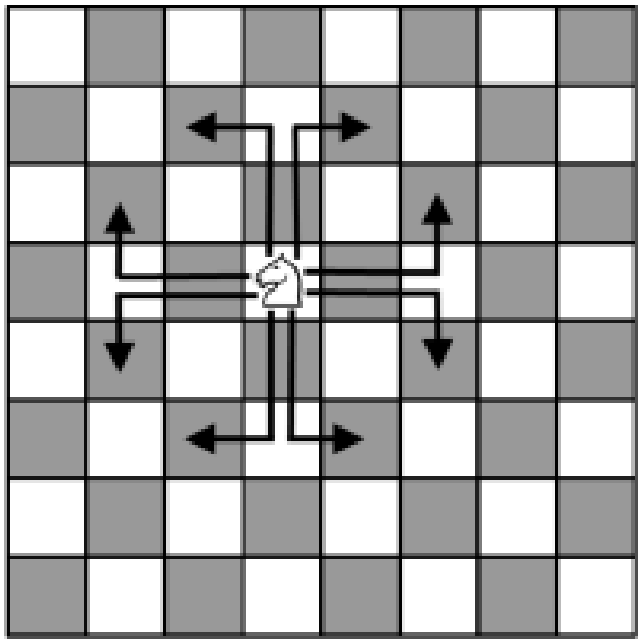
Python

```python
def f (n):
   s = 0
   while n:
      n -= 1
      s += 1 + f(n)
   return s
```

5. **For Computing Lab students only:** complete the skeletons provided for recursive implementations of the *mergesort* and *quicksort* algorithms.

Problems – I

1. In chess, a knight can move to a cell that is horizontally 2 cells and vertically 1 cell away, or vertically 2 cells and horizontally 1 cell away (see Fig. 1). Thus, any knight on the board can attempt exactly eight moves. Some of these moves will result in the knight moving off the board.



Problems – II

Given an $m \times n$ chess-like board, and a knight at position (i, j) (where $0 \leq i < m$ and $0 \leq j < n$), compute the probability that the knight remains on the board after a given number (say k) of steps, assuming that, for a knight on the board, all eight moves are equally likely, but once the knight moves off the board, it has *no valid* moves.

Input Format

$m\ n\ i\ j\ k$

Output Format

The probability value rounded up to 6 decimal places.

Sample Input 0

1 2 0 1 1

Sample Output 0

0.000000

Sample Input 1

8 8 3 3 1

Problems – III

Sample Output 1

1.000000

Sample Input 2

6 6 0 0 1

Sample Output 2

0.250000

Sample Input 3

4 4 0 0 2

Sample Output 3

0.125000

Explanation for the above

	0	1	2	3
0	0/2/2		2	
1			1	2
2	2	1		
3		2		2/2

The knight starts at the top-left corner (0,0). After 1 move, it may either move off the board (permanently), or go to the squares marked 1 (in red / blue). After another move from the square marked with a red (respectively, blue) 1, it lands on one of the squares marked with a red (respectively, blue) 2, or moves off the board. The knight stays on the board for 8 distinct sequences of moves (of 2 steps each). The total number of such moves of 2 steps each is $8 \times 8 = 64$.

Problems – V

The following simplified version of the above problem may help understand how the probability is to be calculated.

Consider a 2×1 grid, with an **X** on the left square. Now toss a fair coin: if it comes up heads, move **X** one square to the right; if it comes up tails, move **X** one square to the left. What is the probability that **X** stays within the grid after k coin tosses?

Solution: The only sequence that permits **X** to stay on the board is HTHTHT. . . Thus, the required probability is $\frac{1}{2^k}$.

If it helps your understanding, you could calculate the probability of **X** moving off the board. This corresponds to the following mutually exclusive sequences:

T, HH, HTT, HTHH, HTHTT, HTHTHH, . . .

The prob. of getting one of these sequences is $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^8} + \cdots + \frac{1}{2^k} = 1 - \frac{1}{2^k}$.

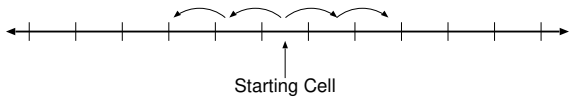
Note that, for this problem, the sequences T and T$Y_2 Y_3 \ldots Y_k$ (where Y_i is either H or T) *have the same probability*. Thus, subsequent coin tosses do not affect the probability of the event that moves **X** off the grid.

Problems – VI

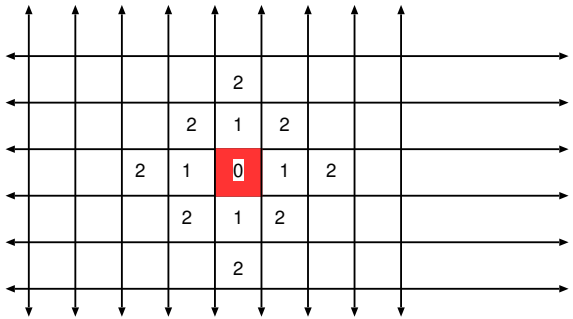
2. Write a program that takes a positive integer $n \leq 5$, and a non-negative integer $k \leq 100$ as command-line arguments, and computes the total number of cells reachable from any starting cell within an infinite, n-dimensional grid in k steps or less. See the examples below. You are only permitted to travel in a direction that is parallel to one of the grid lines; diagonal movement is not permitted.

   NOTE: If you are interested, you may compute a closed-form expression for the required number, but **DO NOT** use the closed-form expression to compute the answer to this problem.

Problems – VII



$n = 1, k = 2$; number of cells reachable = 5



$n = 2, k = 2$; number of cells reachable = 13

For $n = 3$, your output for $k = 0, 1$ and 2 should be 1, 7, and 25, respectively.

3. `http://cse.iitkgp.ac.in/~abhij/course/lab/PDS/Spring15/A4.pdf`